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A NEW PROOF THAT TEICHMÜLLER SPACE IS A CELL

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ABSTRACT. A new proof is given, using the energy of a harmonic map, that Teichmüller space is a cell.

In [**2**] the authors developed a new approach to Teichmüller's famous theorem on the dimension of the unramified moduli space for compact Riemann surfaces. Teichmüller's theorem states (roughly) that the space $\mathcal{T}$ of conformally inequivalent Riemann surfaces of genus p, $p > 1$ (with some topological restrictions) is homeomorphic to Euclidean $\mathbf{R}^{6p-6}$. In proving homeomorphism Teichmüller had put a complete Finsler metric on this space. In [**2**] we showed that $\mathcal{T}$ naturally carried the structure of a C^∞ connected and simply connected differentiable manifold of dimension $6p-6$. The proof of this was straightforward and used only splitting results for symmetric tensors and a standard existence theorem in elliptic partial differential equations. Using somewhat deeper results from the theory of harmonic functions between Riemannian manifolds and a result of Earle and Eells, we were then able to show that our moduli space $\mathcal{T}$ was a contractible manifold.

The purpose of this note is to show that there is a straightforward proof that our Teichmüller space is diffeomorphic to $\mathbf{R}^{6p-6}$. This completes the program of giving the main classical results of Teichmüller strictly in terms of concepts from Riemannian geometry as was formulated in [**2**, **3**, **4**].

1. A quick review of the Fischer-Tromba approach to Teichmüller theory. Let M be a compact oriented surface without boundary. Let $\mathcal{C}$ denote the space of complex structures compatible with the given orientation, $\mathcal{D}$ the space of C^∞ diffeomorphisms, $\mathcal{D}_0$ those homotopic (and hence isotopic) to the identity, and $\mathcal{M}_{-1}$ those Riemannian metrics on M with Riemann scalar curvature negative one. If $c = \{\varphi_i, U_i\}$, $\bigcup U_i = M$, is a complex coordinate atlas for M and $f \in \mathcal{D}$, then $\{\varphi_i \circ f, f^{-1}(U_i)\}$ is a complex coordinate atlas for M which we designate as f^*c. If $g \in \mathcal{M}_{-1}$, then for each $x \in M$, $g(x)\colon T_xM \times T_xM \to R$ is a positive definite symmetric quadratic form on M. By f^*g we mean the form $g(f(x))(df(x)\cdot, df(x)\cdot)$.

One can then form the quotient spaces $\mathcal{M}_{-1}/\mathcal{D}, \mathcal{M}_{-1}/\mathcal{D}_0, \mathcal{C}/\mathcal{D}, \mathcal{C}/\mathcal{D}_0$. The main result of [**2**] is

THEOREM 1.1. *The spaces $\mathcal{T} = \mathcal{M}_{-1}/\mathcal{D}_0$ and $\mathcal{C}/\mathcal{D}_0$ naturally have the structure of a C^∞ connected and simply connected finite-dimensional manifold of dimension*

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$6p-6$. *Moreover there is a naturally defined equivariant diffeomorphism from* $\mathcal{M}_{-1}$ *to* $\mathcal{C}$ *which passes to a diffeomorphism of* $\mathcal{M}_{-1}/\mathcal{D}_0$ *with* $\mathcal{C}/\mathcal{D}_0$. *The space* $\mathcal{C}/\mathcal{D}_0 \cong \mathcal{M}_{-1}/\mathcal{D}_0$ *is the Teichmüller space of* M.

We should remark that the true Riemann space of moduli $R = \mathcal{M}_{-1}/\mathcal{D} = \mathcal{C}/\mathcal{D}$ is not a smooth manifold but does have the structure of an algebraic variety.

For purposes of exposition we wish to describe how one puts a differentiable structure on $\mathcal{M}_{-1}/\mathcal{D}_0$ and to see what the natural tangent space is to this manifold. To see how the diffeomorphism between $\mathcal{M}_{-1}/\mathcal{D}_0$ and $\mathcal{C}/\mathcal{D}_0$ is constructed the reader is referred to [**3**]. Let us think of $\mathcal{M}_{-1}$ as an infinite dimensional submanifold of the space of C^∞ symmetric two tensors S_2 on M. For $g \in \mathcal{M}_{-1}$, the tangent space $T_g\mathcal{O}_g$ to the orbit $\mathcal{O}_g$ of $\mathcal{D}_0$ at g consists of all symmetric tensors of the form $L_X g$, the Lie derivative of g with respect to some vector field X on M. In this case X will be uniquely determined by $h \in T_g\mathcal{M}_{-1}$. The next splitting result of symmetric tensors is basic to our theory.

THEOREM 1.2. *Every* $h \in T_g\mathcal{M}_{-1}$ *can be expressed uniquely as a direct sum* $h = h^{TT} + L_X g$ *where* h^{TT} *is a symmetric two tensor on* M *which is trace free and divergence free. This implies that in a conformal coordinate system (with respect to the metric* g), h^{TT} *has a local representation as*

$$h^{TT} = u\,dx^2 - u\,dy^2 - 2v\,dx\,dy = \mathrm{Re}\{(u+iv)(dx + i\,dy)^2\}$$

where $u+iv$ *is a holomorphic function of the local coordinates* $z = x+iy$, *and* Re *designates the real part.*

Thus every $h \in T_g\mathcal{M}_{-1}$ can be expressed uniquely as a direct sum

$$h = \mathrm{Re}(\xi(z)\,dz^2) + L_X g \tag{1}$$

where $\xi(z)\,dz^2$ is a holomorphic quadratic differential on M with respect to the complex structure induced by g. Moreover every such holomorphic quadratic differential occurs in decomposition (1).

Now the C^∞ manifold structure on $\mathcal{M}_{-1}/\mathcal{D}_0$ follows as a consequence of fact that $\mathcal{D}_0$ acts freely and that as a consequence of the theorem of Riemann-Roch the space of holomorphic quadratic differentials on M has finite dimension $6p-6$.

We summarize these facts as

THEOREM 1.3. *The tangent space to the manifold* $\mathcal{M}_{-1}/\mathcal{D}_0$ *at an element* $[g] \in \mathcal{M}_{-1}/\mathcal{D}_0$ *can be naturally identified with those symmetric two tensors which are trace free and divergence free and also (by taking real parts) with the holomorphic quadratic differentials on* M, *holomorphic with respect to the complex structure induced by* g.

As we already stated, we are viewing $\mathcal{M}_{-1}$ as a differentiable submanifold of the space of all symmetric tensors S_2. There is a natural "weak" L_2 Riemannian structure $\langle\langle\ ,\ \rangle\rangle$ on $\mathcal{M}_{-1}$, $\langle\langle\ ,\ \rangle\rangle_g \colon T_g\mathcal{M}_{-1} \times T_g\mathcal{M}_{-1} \to R$ defined by

$$\langle\langle h^1, h^2\rangle\rangle_g = \int_M h^1 \cdot h^2\, d\mu(g) \tag{2}$$

where, m local coordinates,

$$h^1 \cdot h^2 = g^{ab} g^{cd} h^1_{ac} h^2_{bd}$$

and where $\{g^{ab}\}$ denotes the local representation of the inverse to the matrix $\{g_{ij}\}$ of g, $d\mu(g)$ is the volume element of g, and where the Einstein summation convention is used. One can also give an intrinsic formulation of (2) avoiding local coordinates, as follows.

Using the metric g we can transform h^1, h^2 into $1:1$ tensors H^1, H^2 satisfying

$$g(x)(H_x^i X_x, Y_x) = h^i(x)(X_x, Y_x), \qquad i = 1, 2,$$

for all $X_x, Y_x \in T_xM$. Then each H^i is symmetric with respect to g and for $x \in M$ the trace $\operatorname{tr}(H_x^1 H_x^2)$ is a well-defined function (of x) on M. Then (2) is equivalent to

$$(2') \qquad \langle\langle h^1, h^2 \rangle\rangle_g = \int_M \operatorname{tr}(H^1 H^2)\, d\mu(g).$$

This L_2-Riemannian metric is $\mathcal{D}$ invariant, a fact which follows immediately from the change of variables formula. Thus $\mathcal{D}$ acts on $\mathcal{M}_{-1}$ as a group of isometries.

The important remark is that (1) is an L_2-orthogonal decomposition.

2. Dirichlet's functional on Teichmüller space. Let $g_0 \in \mathcal{M}_{-1}$ and $[g_0]$ denote its class in $\mathcal{M}_{-1}/\mathcal{D}_0$. This fixed g_0 will act as our base point. Let $g \in \mathcal{M}_{-1}$ be any other metric and let $s\colon M \to M$ be viewed as a map from (M, g) to (M, g_0). Using the metrics g and g_0 one defines Dirichlet's energy functional

$$(3) \qquad E_g(s) = \frac{1}{2} \int_M |ds|^2\, d\mu(g)$$

where $|ds|^2 = \operatorname{trace}_g ds \otimes ds$ depends on both metrics g and g_0, and again $d\mu(g)$ is the volume element induced by g.

We may assume that (M, g_0) is isometrically embedded in some Euclidean $\mathbf{R}^k$, which is possible by the Nash-Moser embedding theorem. Thus we can think of $s\colon (M, g) \to (M, g_0)$ as a map into $\mathbf{R}^k$ with Dirichlet's integral having the equivalent form

$$(3') \qquad E_g(s) = \frac{1}{2} \sum_{i=1}^{k} \int_M g(x) \langle \nabla_g s^i(x), \nabla_g s^i(x) \rangle\, d\mu(g).$$

For fixed g, the critical points of E are then said to be *harmonic maps.* From [**1**, **5** and **8**] we have the following result.

THEOREM 2.1. *Given metrics g and g_0 there exists a unique harmonic map $s(g)\colon (M, g) \to (M, g_0)$. Moreover $s(g)$ depends differentiably on g in any H^r topology, $r > 2$, and $s(g)$ is a C^∞ diffeomorphism.*

Consider the function $g \to E_g(s(g))$. This function on $\mathcal{M}_{-1}$ is $\mathcal{D}$-invariant and thus can be viewed as a function on Teichmüller space. To see this one must show that $E_{f^*g}(s(f^*(g))) = E_g(s(g))$. Let $c(g)$ be the complex structure associated to g given by Theorem 1.1. For $f \in \mathcal{D}_0$, $f\colon (M, f^*c(g)) \to (M, c(g))$ is a holomorphic map, and consequently since the composition of harmonic maps and holomorphic maps is still harmonic, we may conclude, by uniqueness, that $S(f^*g) = S(g) \circ f$. Since Dirichlet's functional is invariant under complex holomorphic changes of coordinates, it follows immediately that

$$E_{f^*(g)}(s(g) \circ f) = E_g(s(g)).$$

Consequently for $[g] \in \mathcal{M}_{-1}/\mathcal{D}_0$, define the C^∞ smooth function $\tilde{E}\colon \mathcal{M}_{-1}/\mathcal{D}_0 \to R$ by $\tilde{E}([g]) = E_g(s(g))$. We wish now to prove the main theorem of this note, namely

THEOREM 2.2. *Teichmüller space $\mathcal{M}_{-1}/\mathcal{D}_0$ is C^∞ diffeomorphic to $\mathbf{R}^{6p-6}$.*

To prove this result it suffices to show that $\tilde{E}$ has the following properties:

(i) $\tilde{E}$ is a proper map, i.e. the inverse image of bounded sets in $\mathbf{R}$ under $\tilde{E}$ is compact in $\mathcal{M}_{-1}/\mathcal{D}_0$.

(ii) $[g_0]$ is the only critical point of $\tilde{E}$.

(iii) $[g_0]$ is a nondegenerate minimum.

Once (i) through (iii) are established the result follows immediately from the application of the well-known gradient deformations of Morse theory.

The proof of (i) follows from ideas due to Mumford, Schoen and Yau [**7**], and a result on equicontinuity of harmonic maps (Jost [**6**, p. 20]). Using a result of Mumford, Schoen and Yau show that $E\colon \mathcal{M}_{-1}/\mathcal{D}_0 \to R$ is proper; that is, E is proper on the true space $\mathcal{R}$ of Riemann moduli.

Now suppose that $\tilde{E}[g_n]$ is a bounded sequence. It then follows from [**7**] that $\{g_n\}$ represents a sequence of a class of metrics in $\mathcal{M}_{-1}/\mathcal{D}_0$ all of whose injectivity radii are strictly bounded below. By a version of Mumford's theorem due to Tomi-Tromba [**10**], it follows that there is a subsequence of g_n, call it again g_n and a sequence of diffeomorphisms $f_n \in \mathcal{D}$ such that $f_n^* g_n$ converges.

Let $\gamma_n = f_n^* g_n; r_n = s_n \circ f_n$. Then $E(\gamma_n, r_n)$ is a bounded sequence of real numbers, the γ_n all have injectivity radii strictly bounded below, and $r_n\colon (M, \gamma_n) \to (M, g_0)$ is harmonic. We claim that one can find a subsequence f_n all of which are in the same homotopy class.

Suppose not. Then there is a subsequence of the f_n all in distinct homotopy classes. Again call the subsequence f_n. From Jost's result, the $r_n = s_n \circ f_n$ are equicontinuous. Since the s_n are all homotopic to the identity, this gives a contradiction.

Thus we may assume the f_n are in one homotopy class, $f_n = h_n \circ f$, $f \in \mathcal{D}$ fixed and $h_n \in \mathcal{D}_0$. Then necessarily $h_n^* g_n$ (or more simply $[g_n]$) converges. This proves properness on $\mathcal{M}_{-1}/\mathcal{D}_0$.

To show (ii), again let $s = s(g)\colon (M, g_0) \to (M, g_0)$ be the unique harmonic map determined by g and g_0. Let $\mathcal{N}_g(z)\, dz^2$ be the quadratic differential defined by

$$\mathcal{N}_g(z)\, dz^2 = \sum_{i=1}^{k} \frac{\partial s^i}{\partial z} \cdot \frac{\partial s^i}{\partial z}\, dz^2$$

where s^i is the ith component function of $s\colon (M, g) \to (M, g_0) \hookrightarrow \mathbf{R}^k$, and $z = x + iy$ are local conformal coordinates on (M, g). We next prove

THEOREM 2.3. *$\mathcal{N}_g(z)\, dz^2$ is a holomorphic quadratic differential on $(M, c(g))$.*

PROOF. Let Ω denote the second fundamental form of $(M, g_0) \subset \mathbf{R}^k$. Thus for each $p \in M$, $\Omega(p)\colon T_pM \times T_pM \to T_pM^\perp$. Let Δ denote the Laplacian maps from (M, g) to (M, g_0), and Δ_β denote the Laplace-Beltrami operator on functions. Then if s is harmonic we have

$$0 = \Delta s = \Delta_\beta s + \sum_{j=1}^{2} \Omega(s)(ds(e_j),\, ds(e_j)) \tag{4}$$

with $e_1(p), e_2(p)$ an orthonormal basis for T_pM (w.r.t. the matrix g). $\mathcal{N}_g$ will be holomorphic if

$$\frac{\partial}{\partial \bar{z}}\left(\sum_{i=1}^{k} \frac{\partial s^i}{\partial z}\cdot\frac{\partial s^i}{\partial z}\right) = 0.$$

But this equals

$$2\sum_{i=1}^{k} \Delta_\beta s^i \cdot \frac{\partial s^i}{\partial z}$$

and by (4) we see that this in turn equals

$$\begin{aligned} &-2\sum_{i=1}^{k}\sum_{j=1}^{2} \Omega^i(s)\,(ds(e_j),\, ds(e_j))\cdot\frac{\partial s^i}{\partial z} \\ &\quad = -2\sum_{i=1}^{k}\sum_{j=1}^{2}\left\{\sum \Omega^i(s)\left(ds(e_j),\, ds(e_j)\frac{\partial s^i}{\partial x}\right) + i\Omega(s)\left(ds(e_j),\, ds(e_j)\frac{\partial s^i}{\partial y}\right)\right\}. \end{aligned}$$

Since $\Omega(p)$ takes value in $T_pM^\perp$ it follows that both the real and the imaginary parts of this expression vanish. □

From 1.2 we saw that $\xi = \mathrm{Re}(\mathcal{N}_g(z)\,dz^2)$ is a trace free divergence free symmetric two tensor on (M, g). Let $\rho \in T_{[g]}\mathcal{M}_{-1}/\mathcal{D}_0$. We know from 1.3 that we may think of ρ as a trace free divergence free symmetric two tensor. From [**10**] we have the following result:

THEOREM 2.4. $D\tilde{E}([g])\rho = -\langle\langle \xi, \rho\rangle\rangle_g$. *Thus* $[g]$ *is a critical point of* $\tilde{E}$ *if* $\rho = 0 \equiv \mathrm{Re}(\mathcal{N}_g(z)\,dz^2)$, *or if* $\mathcal{N}_g(z)\,dz^2 \equiv 0$.

THEOREM 2.5. $\mathcal{N}_g(z)\,dz^2 = 0$ *implies that* $[g] = [g_0]$.

PROOF. $\mathcal{N}_g(z)\,dz^2 = \{|s_x|^2 - |s_y|^2 + 2i\langle s_x, s_y\rangle\}\,dz^2$. Thus $\mathcal{N}_g(z)\,dz^2$ implies that s is weakly conformal. Since s is a diffeomorphism it is conformal. Thus $s\colon (M, c(g)) \to (M, c(g_0))$ is holomorphic and hence $[g] = [g_0]$.

It remains to show (iii). It is clear that since $\mathcal{N}_g(z)\,dz^2 \equiv 0\,(s(g_0) = \mathrm{id})$ that $[g_0]$ is a critical point.

Let $\rho, \nu \in T_{[g_0]}\mathcal{M}_{-1}/\mathcal{D}_0$ be trace free, and divergence free symmetric two tensors. Then a straightforward computation yields

THEOREM 2.6. *The second derivative or Hessian of* $\tilde{E}$ *at* $[g_0]$ *is given by the formula*

$$D^2\tilde{E}([g_0])(\rho, \nu) = 2\int_M \rho\cdot\nu\; d\mu(g_0) = 2\langle\langle \rho, \nu\rangle\rangle_{g_0}.$$

Thus the Hessian of $\tilde{E}$ *at* $[g_0]$ *is essentially the natural inner product on* $T_{[g_0]}\mathcal{M}_{-1}/\mathcal{D}_0$ *and hence a positive definite quadratic form. This concludes the proof of our main result* 2.2.

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